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Ill-Defined Block-Spin Transformations at Arbitrarily High Temperatures

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Examples are presented of block-spin transformations which map the Gibbs measures of the Ising model in two or more dimensions at temperature intervals extending to arbitrarily high temperatures onto non-Gibbsian measures. In this way we provide the first example of this kind of pathology for very high temperatures, and as a corollary also the first example of such a pathology happening at a critical point.

KEY WORDS: Non-Gibbsian measures; block-spin map pathology.

1. INTRODUCTION

In refs. 27, 28, and 26 it was shown how various renormalization-group (RG) maps acting on Gibbs measures produce non-Gibbsian measures. In physicists' language, this means that a "renormalized Hamiltonian" cannot be defined. The examples presented there were valid at low temperatures or close to a first-order phase transition. The underlying mechanism—pointed out first by Griffiths, Pearce, and Israel[7, 8, 10]—is the fact that for the constraints imposed by particular choices of block-spin configurations, the resulting system exhibits a first-order phase transition. For this to happen, it was expected that the original system should be itself at or in the vicinity of a phase transition. Block-average and majority-rule transformations, however, provided counterexamples to this belief, in that they lead to non-Gibbsianness for arbitrarily large values of the magnetic field (at low temperatures).[28, 26]

On the other hand, there exist complementary results, either proving or providing heuristic but plausible arguments, that renormalized Gibbsian measures will be Gibbsian again at high fields or high temperatures[7, 8, 10, 2, 11],

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in the uniqueness region after a sufficiently often iterated decimation (possibly combined with another transformation)[18,19] and in intervals around and including critical points for a rather general class of transformations.[12,1,29,4,13]

For further investigations about the non-Gibbsianness of various measures of interest in statistical mechanics, see ref. 28 and references quoted there, and also for more recent results refs. 16, 6, 24, 23, 17, 30, and 15.

Here we present the first examples where such a Griffiths–Pearce–Israel-type pathology occurs in the high-temperature region within the domain of (strong) complete analyticity.[5] Our example also includes a critical temperature pathology. Thus we show that there are no pathology-free domains for general block-spin maps. Note that the Potts decimation example of ref. 26 provides a weaker statement; although a pathology occurs above the transition temperature strong complete analyticity does not apply, even though, at least in the two-dimensional case, one expects that the weak complete analyticity property (for sufficiently regular volumes) holds.[20–22,25]

2. THE EXAMPLE

We consider the standard nearest neighbor Ising model with (formal) Hamiltonian

$$H = \sum_{\langle i,j \rangle} -\sigma(i)\,\sigma(j) \tag{2.1}$$

at inverse temperature β.

We consider the following transformation: Divide the lattice Z^d into blocks B_j^L of linear size L and define block-spins $\sigma'(j)$ by

$$\sigma'(j) = \sum_{i \in B_j^L} \sigma(i) \tag{2.2}$$

except in the case where all spins in the block are minus, in which case

$$\sigma'(j) = +L^d \tag{2.3}$$

This induces a map T_L on the measures on the space of spin-configurations. For more details we refer to ref. 28.

Now we have the following theorem:

Theorem. Let μ_β be the Gibbs measure of the nearest neighbor Ising model in dimension $d \geqslant 2$, at inverse temperature $\beta > 0$. Then there exists a constant L_0 such that for all $L > L_0$, $\mu'_L = \mu \circ T_L$ is non-Gibbsian.

Proof. The proof follows the scheme of ref. 28, Section 4.2. The special block-spin configuration which will be the point of (essential) discontinuity for a suitably chosen conditional probability will be the choice $\sigma'(j) = L^d$ in each block. This means that within each block all spins are unanimous, either in the plus or in the minus direction. Thus, imposing the special block-spin configuration as a constraint means that the sum of all spins in each block can take only two values (either unanimously positive or unanimously negative). Hence the thus-constrained system is again a nearest neighbor Ising model, but now at an inverse temperature $\beta' = L^{d-1} \times \beta$. For L large enough, β' becomes so large that the constrained system is in the phase transition (low-temperature) regime (step 1 of the proof). The block-spin boundary conditions selecting the plus and minus measures are $\sigma'(j) = +$ or $-(L^d - 2)$ in each boundary block. This means that in every block there is only one spin opposing all the others. When $L > 2$, a standard Peierls argument proves the phase transition (step 2 of the proof). As the expectation of $\sigma'(0)$ (unfixed according to step 3 of the proof) is different in the plus and minus phases, we see that the scheme of proof of ref. 28, Section 4.2, can be straightforwardly applied.

3. COMMENTS AND CONCLUSIONS

1. As a corollary of the theorem we obtain the first example where a Griffiths–Pearce–Israel pathology occurs *at* a critical point.

2. The block-spin map defined above is admittedly artificial from a physical point of view. However, the heuristic considerations of refs. 1 and 4, which were designed for the block-average transformation, formally equally well apply to this map, but lead to an incorrect conclusion. This illustrates the danger of relying on formal as opposed to fully rigorous arguments.

3. If we consider the sequence $\mu \circ T_L$, as a corollary to the (local) central limit theorem,[9,3,2] it is easy to see that, with a proper rescaling of the block spins, it converges weakly to an independent Gaussian zero-mean measure. The measures in the sequence are almost all (that is, all except for possibly a finite number) non-Gibbsian.

4. Applying a decimation transformation often enough will make the measure Gibbsian again.[19] This seems to mean that there are maps which reinforce, as well as maps which weaken, the Gibbsianness of measures in the uniqueness region. What the behavior of a "typical" map—whatever that might be—is remains open. In this sense, how robust the phenomenon of renormalization-group pathologies is remains an issue of debate. Our result

shows that the pathologies are more widespread than was known before, while the results of refs. 18, 19, and 1 go in the opposite direction, indicating how one might make some of these pathologies disappear.

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